

Dissecting the Hadronic Contributions to $(g-2)_\mu$ by Schwinger's Sum RuleFranziska Hagelstein^{1,2} and Vladimir Pascalutsa¹¹*Institut für Kernphysik & Cluster of Excellence PRISMA, Johannes Gutenberg-Universität Mainz, D-55128 Mainz, Germany*²*Albert Einstein Center for Fundamental Physics, Institute for Theoretical Physics, University of Bern, CH-3012 Bern, Switzerland*

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The theoretical uncertainty of $(g-2)_\mu$ is currently dominated by hadronic contributions. In order to express those in terms of directly measurable quantities, we consider a sum rule relating $g-2$ to an integral of a photoabsorption cross section. The sum rule, attributed to Schwinger, can be viewed as a combination of two older sum rules: Gerasimov-Drell-Hearn and Burkhardt-Cottingham. The Schwinger sum rule has an important feature, distinguishing it from the other two: the relation between the anomalous magnetic moment and the integral of a photoabsorption cross section is linear, rather than quadratic. The linear property makes it suitable for a straightforward assessment of the hadronic contributions to $(g-2)_\mu$. From the sum rule, we rederive the Schwinger $\alpha/2\pi$ correction, as well as the formula for the hadronic vacuum-polarization contribution. As an example of the light-by-light contribution, we consider the single-meson exchange.

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Introduction.—The anomalous magnetic moment (AMM) of the muon, $\kappa_\mu \equiv 1/2(g-2)_\mu$, serves as a stringent precision test of the Standard Model (SM). And at present, it does not work out for the SM—the experimental value is about 3σ away from the SM prediction [1,2]. While the uncertainties of the SM and the experimental value are comparable, the new Fermilab experiment [3,4] will, in a few years, reduce the experimental error bar by nearly a factor of 4. The prospects for reducing the SM (theory) uncertainty are, on the other hand, more obscure. The SM error bar is dominated by the *hadronic contributions*, which are very difficult to compute in the SM due to the nonperturbative nature of quantum chromodynamics (QCD). In the present SM value, these contributions are determined empirically, using general relations to other experimental observables in combination with mesonic model calculations, rather than from QCD directly. It is the necessity of resorting to models—particularly in evaluation of the so-called hadronic light-by-light (HLbL) contributions [cf. Fig. 1(b)]—which makes it difficult to reduce the uncertainty of the current SM value.

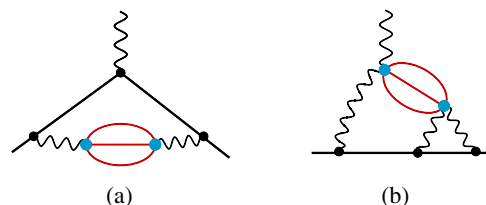
In the future, lattice QCD will deliver a sufficiently precise *ab initio* calculation of the HLbL contribution; for recent progress in this direction, see Refs. [5–8]. Until then, the best hope for improvement is to replace the model evaluations with model-independent, “data-driven”

approaches based on dispersion theory. The data-driven approach is fairly well founded and routinely used for the hadronic vacuum polarization (HVP) contribution [Fig. 1(a)], since it can be written exactly as a dispersion integral of the decay rate of a virtual timelike photon into hadrons, which to a good approximation is expressed in terms of the observed ratio $\mu^+\mu^-/e^+e^- \rightarrow \text{hadrons}$; see, e.g., Refs. [9,10]. The HLbL contribution is much more complicated from this point of view, because it involves the dispersion relations for three- and/or four-point functions, rather than for a two-point function as in the case of HVP; see Refs. [11–14] and [15] for the two recent approaches to this problem.

Here we consider yet another approach to a data-driven evaluation of hadronic contributions rooted in dispersion theory. It is based on sum rules for Compton scattering, of which a famous example is the Gerasimov-Drell-Hearn (GDH) sum rule [16–18]:

$$\frac{\alpha}{m^2} \kappa^2 = \frac{1}{2\pi^2} \int_{\nu_0}^{\infty} d\nu \frac{\sigma_{3/2}(\nu) - \sigma_{1/2}(\nu)}{\nu}. \quad (1)$$

On the left-hand side (lhs), we have $\alpha = e^2/4\pi \simeq 1/137$, the fine-structure constant, and κ , the AMM of the spin-1/2

FIG. 1. Hadronic contributions to $(g-2)_\mu$: (a) HVP, (b) HLbL.

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target particle with mass m , whereas the rhs contains the helicity-difference cross section of total photoabsorption on that particle, integrated over the photon energy ν , starting from the photoabsorption threshold ν_0 .

This is the sort of relation we are looking for: the cross sections can in principle be measured in hadronic channels separately (e.g., $\gamma\mu \rightarrow \pi^0\mu$), and hence we can “measure” the hadronic contributions to κ . Unfortunately, this strategy would not work here, because the sum rule involves κ^2 and thus we would be probing a very tiny number—recall that the hadronic contribution to κ_μ is of the order 10^{-8} . In powers of α , the hadronic contribution to κ_μ starts at $O(\alpha^2)$; therefore, the lhs of the GDH sum rule is $O(\alpha^5)$, whereas the cross sections of hadronic photoproduction starts at $O(\alpha^3)$. This means there is a huge (at least 5 orders of magnitude) cancellation under the GDH integral, and therefore these cross sections would need to be measured with unprecedented accuracy.

The same κ^2 feature prevents this sum rule from being useful in theoretical calculations: to compute κ to $O(\alpha^n)$, one needs to know the cross sections to $O(\alpha^{2n+1})$, which in fact is a more difficult calculation. This was explicitly demonstrated by Dicus and Vega [19], who reproduced the Schwinger’s correction ($\alpha/2\pi$) through the GDH sum rule. Taking a derivative of the GDH sum rule with respect to κ linearizes the sum rule and hence simplifies the calculations [20,21]. The drawback of the GDH-derivative method is that the rhs loses a direct connection to experimental observables: the helicity-difference cross section is replaced by a derivative quantity which cannot be accessed in experiment.

Therefore, in what follows we focus on a sum rule which is linear in the AMM and involves an observable cross-section quantity.

The Schwinger sum rule.—Consider the following relation, referred to as the Schwinger sum rule [22,23]:

$$\kappa = \frac{m^2}{\pi^2\alpha} \int_{\nu_0}^{\infty} d\nu \left(\frac{\sigma_{\text{LT}}(\nu, Q^2)}{Q} \right)_{Q^2=0}, \quad (2)$$

where $\sigma_{\text{LT}}(\nu, Q^2)$ —the longitudinal-transverse photoabsorption cross section—is an observable (response function) corresponding to an absorption of a polarized virtual photon with energy ν and spacelike virtuality Q^2 on the target with mass m and AMM κ , whereby the spin of the target flips. This response function is rather common in the studies of nucleon spin structure via electron scattering. For instance, it plays the central role in the evaluation of the so-called δ_{LT} polarizability of the proton, and hence in the “ δ_{LT} puzzle” (cf. Ref. [24] for a recent review). One can introduce it for the muon as well, and benefit from the fact that the sum rule is linear in κ , rather than quadratic. However, before applying it to the muon case, let us briefly see how it comes about.

The Schwinger sum rule can be viewed as a consequence of the Burkhardt-Cottingham (BC) and GDH sum rules, in fact, a linear combination of those. Introducing the spin-structure functions $g_1(x, Q^2)$ and $g_2(x, Q^2)$ of the spin-1/2 target, with $x = Q^2/2m\nu$ being the Bjorken variable, the BC sum rule reads as follows [25]: $\int_0^1 dx g_2(x, Q^2) = 0$. Separating the structure functions into the parts accessed in elastic and inelastic electron scattering, the elastic part is expressed in terms of the Dirac and Pauli form factors, $F_1(Q^2)$ and $F_2(Q^2)$:

$$g_1^{\text{el}}(x, Q^2) = \frac{1}{2} F_1(Q^2) [F_1(Q^2) + F_2(Q^2)] \delta(1-x), \quad (3a)$$

$$g_2^{\text{el}}(x, Q^2) = -\frac{Q^2}{8m^2} F_2(Q^2) [F_1(Q^2) + F_2(Q^2)] \delta(1-x), \quad (3b)$$

whereas the inelastic part, $\bar{g}_i = g_i - g_i^{\text{el}}$, can be expressed in terms of the response functions σ_{LT} and $\sigma_{\text{TT}} \equiv 1/2(\sigma_{1/2} - \sigma_{3/2})$; for more details, see, e.g., Sec. 5.2 of Ref. [24]:

$$\bar{g}_1(x, Q^2) = \frac{1}{4\pi^2\alpha} \frac{m\nu^3}{\nu^2 + Q^2} \left(\frac{Q}{\nu} \sigma_{\text{LT}} + \sigma_{\text{TT}} \right), \quad (4a)$$

$$\bar{g}_2(x, Q^2) = \frac{1}{4\pi^2\alpha} \frac{m\nu^3}{\nu^2 + Q^2} \left(\frac{\nu}{Q} \sigma_{\text{LT}} - \sigma_{\text{TT}} \right). \quad (4b)$$

In the limit of $Q^2 \rightarrow 0$, with $F_1(0) = 1$ and $F_2(0) = \kappa$, the BC sum rule yields

$$\begin{aligned} (1 + \kappa)\kappa &= \lim_{Q^2 \rightarrow 0} \frac{8m^2}{Q^2} \int_0^{x_0} dx \bar{g}_2(x, Q^2), \\ &= \frac{m^2}{\pi^2\alpha} \int_{\nu_0}^{\infty} d\nu \left(\frac{\sigma_{\text{LT}}}{Q} - \frac{\sigma_{\text{TT}}}{\nu} \right)_{Q^2=0}, \end{aligned} \quad (5)$$

where $x_0 = Q^2/2m\nu_0$ is the inelastic threshold of the Bjorken variable. [Here the explicit use of $F_1(0) = 1$ limits the applicability of the resulting sum rule in Eq. (2) to charged particles, in contrast to, e.g., the GDH sum rule, which holds as well for a neutral particle, such as the neutron.] Now, the GDH sum rule allows us to cancel the κ^2 on the lhs against the σ_{TT} term on the rhs, resulting in Eq. (2). The latter can also be rewritten in terms of the spin-structure functions as

$$\kappa = \lim_{Q^2 \rightarrow 0} \frac{8m^2}{Q^2} \int_0^{x_0} dx [\bar{g}_1 + \bar{g}_2](x, Q^2). \quad (6)$$

Thus, we “only” need to know how (a moment of) the muon spin-structure function combination $g_1 + g_2$ is affected by hadronic contributions.

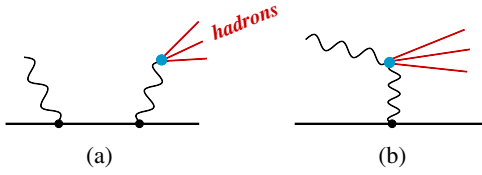


FIG. 2. Mechanisms of hadron photoproduction off a lepton: (a) Timelike Compton scattering (crossed diagram omitted), (b) Primakoff effect.

Let us now examine the hadronic contributions to $(g - 2)_\mu$ using the Schwinger sum rule. The first thing to consider is the hadron production on the muon shown in Fig. 2; i.e., $\gamma\mu \rightarrow \mu + \text{hadrons}$. Examples of these processes are: $\gamma\mu^\pm \rightarrow \pi^0\mu^\pm$, $\gamma\mu^\pm \rightarrow \pi^+\pi^-\mu^\pm$. Here one can distinguish two mechanisms: timelike Compton scattering [Fig. 2(a)], and the Primakoff effect [Fig. 2(b)]. They add up incoherently (i.e., there is no interference term) because of C -parity conservation, viz. Furry's theorem.

Both mechanisms begin to contribute at $O(\alpha^3)$ to σ_{LT} , and hence at $O(\alpha^2)$ to $g - 2$. The first mechanism (timelike) corresponds with the HVP contribution [Fig. 1(a)], and thus provides an alternative access to it; see below. On the other hand, the leading-order contribution of the Primakoff mechanism to $g - 2$ should vanish exactly, as it does not correspond with the HVP contribution, and there is no other hadronic contribution to $g - 2$ at this order. An explicit proof of this statement [i.e., the vanishing effect of Fig. 2(b) on $g - 2$] should be possible through the use of the light-by-light scattering sum rules [26,27]. As a result, the Primakoff mechanism can only contribute in interference with subleading effects, such as the one shown in Fig. 3 for the case of $\pi^0\gamma$ and π^0 production.

The main advantage of using the Schwinger sum rule, however, is that one need not be concerned with computing the subleading effects of hadronic production—they all can in principle be measured experimentally. This can be

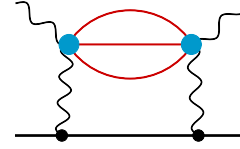


FIG. 4. The HLbL contribution to Compton scattering.

achieved at an electron-muon collider with polarized beams needed to access the spin-structure functions. Tagging is not necessary, since we only need the quasireal-photon limit. No separation of radiative corrections is necessary: as long as hadrons are present in the final state, they are part of the hadronic contributions to the spin-structure functions of one of the leptons. In fixed-target experiments, one would need to measure the recoil electron polarization.

Apart from the above mentioned hadron production channels, the muon structure functions can be affected by hadrons in the loops. The most important (in orders of α) is the effect of HLbL on the Compton scattering (CS), shown in Fig. 4, interfering with the tree-level Compton effect (Fig. 5). Note that here the initial photon is quasireal, whereas the final one is real. Thus, the evaluation of the HLbL contribution to σ_{LT} involves the HLbL amplitude with only two virtual photons. This is substantially simpler than the corresponding HLbL contribution to $g - 2$ shown in Fig. 1(b), which involves the LbL amplitude with three virtual photons and one quasireal.

Another HLbL effect, of the same order in α , arises from the interference of the diagrams in Fig. 6, describing the hadronic contribution to the $\mu\gamma\gamma$ channel. Here the treatment of the HLbL contribution is even simpler than in the Compton channel, since the HLbL amplitude is not in the loop and only one of the four photons is virtual.

Before considering these hadronic contributions further, it is instructive to compute the leading QED contribution of Fig. 5 by itself. A straightforward calculation yields

$$\sigma_{\text{LT}}^{\gamma\mu \rightarrow \gamma\mu}(\nu, Q^2) = \frac{\pi\alpha^2 Q(s - m^2)^2}{4m^3\nu^2(\nu^2 + Q^2)} \left(-2 - \frac{m(m + \nu)}{s} + \frac{3m + 2\nu}{\sqrt{\nu^2 + Q^2}} \text{arccoth} \frac{m + \nu}{\sqrt{\nu^2 + Q^2}} \right), \quad (7)$$

with $s = m^2 + 2m\nu - Q^2$. Substituting this expression into the Schwinger sum rule, one obtains for κ the Schwinger correction: $\alpha/2\pi$. This exercise thus provides a check of the sum rule in leading-order QED, similar to the one done by Tsai *et al.* [28].

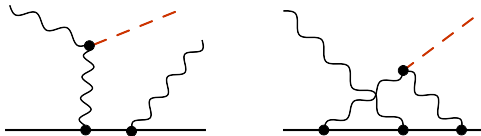


FIG. 3. Subleading mechanisms accompanying the single-meson photoproduction.

Hadronic vacuum polarization.—To reproduce the leading HVP contribution [Fig. 1(a)] through the hadron photoproduction mechanism shown in Fig. 2(a), we factorize the invariant mass distribution $d\sigma(\gamma\mu \rightarrow \mu X)/dM_X^2$, arising from Fig. 2(a), into the cross sections of timelike Compton scattering $\sigma(\gamma\mu \rightarrow \gamma^*\mu)$ and of the subsequent

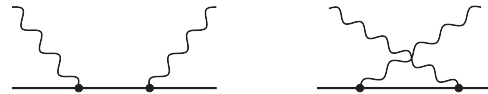


FIG. 5. Tree-level Compton scattering diagrams.

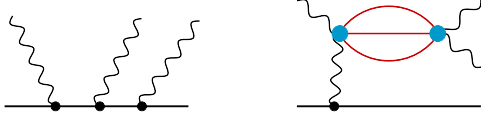
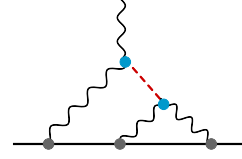


FIG. 6. The HLbL contribution through two-photon production.

photon decay into hadrons $\sigma(\gamma^* \rightarrow X)$. The latter is, by unitarity, expressed via the absorptive part of the hadronic contribution to vacuum polarization, $\text{Im} \Pi^{\text{had}}(q^2)$. The


 FIG. 7. π^0 -exchange HLbL contribution to $(g-2)_\mu$.

tree-level LT cross section of Compton scattering, with initial and final photon virtualities given by $q^2 = -Q^2$ and $q'^2 = M_X^2$, respectively, is easily computed to yield

$$\left[\frac{\sigma_{\text{LT}}^{\gamma\mu \rightarrow \gamma^*\mu}(\nu, Q^2)}{Q} \right]_{Q^2=0} = \frac{\pi\alpha^2}{2m\nu^3} \left[-(5s + m^2 + M_X^2)\lambda + (s + 2m^2 - 2M_X^2) \log \frac{\beta + \lambda}{\beta - \lambda} \right], \quad (8)$$

with $s = m^2 + 2m\nu$, $\beta = (s + m^2 - M_X^2)/2s$, and $\lambda = (1/2s)\sqrt{[s - (m + M_X)^2][s - (m - M_X)^2]}$. From the Schwinger sum rule, we then have

$$\kappa = \frac{m^2}{\pi^2\alpha} \int_{4m_\pi^2}^{\infty} dM_X^2 \int_{\nu_0}^{\infty} d\nu \left[\frac{1}{Q} \frac{d\sigma_{\text{LT}}^{\gamma\mu \rightarrow \mu X}(\nu, Q^2)}{dM_X^2} \right]_{Q^2=0} = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} dM_X^2 \frac{\text{Im} \Pi^{\text{had}}(M_X^2)}{M_X^2} \frac{m^2}{\pi^2\alpha} \int_{\nu_0}^{\infty} d\nu \left[\frac{\sigma_{\text{LT}}^{\gamma\mu \rightarrow \gamma^*\mu}(\nu, Q^2)}{Q} \right]_{Q^2=0}, \quad (9)$$

where $\nu_0 = M_X(1 + M_X^2/2m^2)$ is the photoproduction threshold, while $4m_\pi^2$ is set by the lightest produced state (here, the $\pi^-\pi^+$ pair). Finally, performing the integration over ν , we obtain

$$\frac{m^2}{\pi^2\alpha} \int_{\nu_0}^{\infty} d\nu \left[\frac{\sigma_{\text{LT}}^{\gamma\mu \rightarrow \gamma^*\mu}(\nu, Q^2)}{Q} \right]_{Q^2=0} = \frac{\alpha}{\pi} K(M_X^2/m^2) \equiv \frac{\alpha}{\pi} \int_0^1 dx \frac{x^2(1-x)}{x^2 + (1-x)(M_X^2/m^2)}, \quad (10)$$

and hence the standard expression for the HVP contribution (see, e.g., Ref. [9]),

$$\kappa^{\text{HVP}} = \frac{\alpha}{\pi^2} \int_{4m_\pi^2}^{\infty} ds K(s/m^2) \frac{\text{Im} \Pi^{\text{had}}(s)}{s}, \quad (11)$$

is exactly reproduced.

In practice, a determination of the HVP contribution through the Schwinger sum rule has an important conceptual difference from the standard practice of measuring $e^+e^- \rightarrow \text{hadrons}$. The latter method involves an approximation of the single-photon exchange; the two-photon exchange effects ought to be removed. In the sum-rule method, the two-photon exchange and other subleading effects need not be removed; they are part of the sought hadronic contribution.

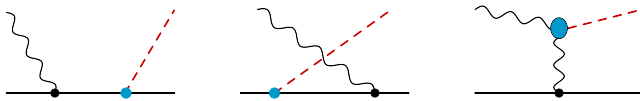


FIG. 8. Single-meson photoproduction off a lepton.

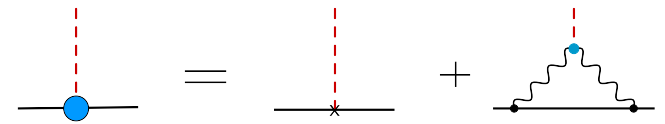


FIG. 9. Pseudoscalar meson coupling to leptons.

Further novel features of calculating the hadronic contributions through the Schwinger sum rule can be seen in the following example of the meson exchange contribution.

Pseudoscalar meson contribution.—The neutral pseudoscalar mesons π^0 and η play a significant role in the HLbL contribution through the mechanism shown in Fig. 7. Let us see how this mechanism is evaluated using the Schwinger sum rule.

In the hadronic channel, we need to know the LT cross section for the single-meson photoproduction off the lepton. This can in principle be measured directly, or calculated to leading order by evaluating the diagrams in Fig. 8. Note that, in addition to the Primakoff mechanism (last diagram), we have here the subleading (in α) mechanisms of the type shown in Fig. 3. The latter mechanisms are effectively represented by the first two graphs in Fig. 8, where the meson-lepton-lepton ($\pi\ell\ell$) coupling is fixed

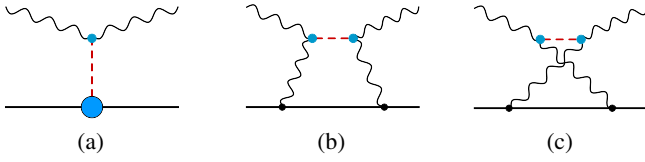


FIG. 10. Pseudoscalar-meson contribution to Compton scattering. Diagram (a) shows t -channel exchange with the $\pi\ell\ell$ form factor given by Fig. 9; (b) and (c) show the direct and crossed box, respectively. Drawn using Jaxodraw [33].

from the decay width of pseudoscalar mesons into leptons (i.e., $\pi^0 \rightarrow e^+e^-$ and $\eta \rightarrow \mu^+\mu^-$).

The same experimental information on pseudoscalar-into-lepton decays fixes the counterterm needed to renormalize the vertex calculation of the form factor in Fig. 9, which is needed in the Compton channel calculation, cf. Fig. 10(a). It is interesting that this $\pi\ell\ell$ form factor enters profoundly in the calculation of π^0 exchange in the hyperfine splitting of muonic hydrogen [29–32]. The π^0 effects in $(g-2)_\mu$ and muonic hydrogen are thus inter-related.

It is important to realize that in calculating the π^0 contribution through the diagrams in Figs. 8 and 10, we do not need the transition from factor (TFF) with two virtual photons. We only need the TFF for a single virtual photon [i.e., $F_{\pi^0\gamma\gamma^*}(Q^2)$] in the box graphs of Figs. 10(b) and 10(c), since the external photons are (quasi)real. The doubly virtual TFFs could be used in evaluation of the $\pi\ell\ell$ form factor in Fig. 9. However, their impact therein is largely diminished by the renormalization and the use of the empirical width.

Conclusion.—We have considered the hadronic contributions (HVP and HLbL) to $g-2$, and showed how they can be assessed using the Schwinger sum rule. The sum rule separates the hadronic contributions into two types: (i) Hadron photoproduction (Fig. 2). (ii) HLbL contribution to nonhadronic channels (Figs. 4 and 6). Type (i) has a clear relation to observables. These are the spin-structure functions of the lepton with hadrons in the final state (hadronic channels) that could in principle be measured in electron-muon collisions. This type of contribution is readily suited for a model-independent, “data-driven” evaluation.

In type (ii), the hadrons may appear in the loops, similar to the sought HLbL contribution to $g-2$ [Fig. 1(b)]. However, the sum rule evaluation only requires the HLbL for the situation of two virtual photons forming a loop, rather than three virtual photons forming two loops as required in the direct evaluation of Fig. 1(b). The former evaluation, therefore, requires much less information about HLbL, and is technically simpler. For example, the evaluation of the neutral pion, and other single-meson contributions, will only require the transition form factor to one real and one virtual photon ($M \rightarrow \gamma\gamma^*$), rather than two virtual photons ($M \rightarrow \gamma^*\gamma^*$), as required usually.

The Schwinger sum rule is thus a very promising tool for a data-driven evaluation of the hadronic contributions to $(g-2)_\mu$. Despite being quite different from the existing dispersive approaches, the present approach may benefit from the dispersive analysis of the HLbL amplitude by Colangelo *et al.* [11–14], trimmed to the narrower kinematical range required for the evaluation of Fig. 4. In a more distant perspective, the type-(ii) contribution will be calculable in lattice QCD.

We have outlined how this program works for the pseudoscalar meson contributions. With very few modifications it applies, of course, to the axion contributions to $g-2$, which have lately been receiving renewed attention in connection with collider searches [34,35].

The advantages of evaluating the axion and other beyond-SM contributions by using the Schwinger sum rule are less obvious than in the hadronic case, where data-driven approaches are generally desirable in the absence of precise *ab initio* calculations. And, even in a more advanced lattice-QCD era, the presented sum-rule approach may be advantageous, if only for its clear-cut separation of the explicit hadron production from the virtual hadronic effects.

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